International Journal of Mathematical Archive-9(3), 2018, 1-3
JMA Available online through www.ijma.info ISSN 2229 – 5046

CONSTRUCTION OF FLUID FLOW USING CONFORMAL MAPPING

MOHAMMED MUKHTAR MOHAMMED ZABIH*
Department of Mathematics,
Maulana Azad College of Arts, Commerce and Science, Rouza Bagh, Aurangabad-431001.

R. M. LAHURIKAR
Department of Mathematics,
Govt. College of Arts And Science, Kile Ark, Aurangabad-431001, (M.S.) INDIA.

(Received On: 23-01-18; Accepted On: 21-02-18)

ABSTRACT

Flow is ubiquitous in nature. Some flows are exquisite, like Karman's vortex street, while some are substantial, like flow induced by Joukowski transformation and some of them are clumsy, like powerful tornadoes left behind aircraft wing. However, the perturbation in flows cannot be neglected. But in all diverse situation it is inevitable to study the flow patterns. A profound contribution is made by the conformal mapping to construct the flows of fluid and tempted a pretty big class of scientist and engineers to work with. One such attempt is made here to use conformal mapping for construction of a flow of an ideal fluid, which may title itself entices to you.

Keywords – *Conformal mapping, Ideal fluid*

1. INTRODUCTION

Conformal mapping [3, 5, 10] is born and brought up in complex analysis, as its roots pervade down it stands firmly and produces a reasonable number of branches which works for the welfare of science and technology e.g. In the field of fluid mechanics, potential theory, electrostatics and cartography etc. Its property of mapping conformally a geometry from $z = x + iy$ plane (domain) into a different geometry in $w = u + iv$ plane (codomain) such that the angle between any two curves is preserved in the transportation from z-plane to w-plane makes it admirable and enhances our ability to use it.

In the problems dealing with the velocity of a particle in a moving fluid in a certain channel and streamlining some obstacles (rigid body) on its way, with the temperature of heated body and with the electric potential at points in space surrounding a charged capacitor, it is needed to evaluate the velocity, temperature and the potential respectively. Such computation can be easily achieved if the streamlined body is simple in shape, for example, in the construction of aircraft wing it is necessary to compute the velocity of the particle of the air flow streamlining the wings of aircraft. Velocity computation is easy when the streamlined body is circular (instead of aircraft wing).

Application of conformal mappings in fluid mechanics can be long way back to the work of Gauss, Riemann, Weierstrass, Neumann and Schwarz. But it is Rayleigh, who for the first time give complete treatment of conformal mapping in aerodynamics. However," the father of Russian aviation" Joukowsky, employed successfully, so called, the Joukowsky transform, to show that the cross section of air craft wing is similar to the cross section of circle [6]. In [8] conformal mapping is applied to the potential theory, through computer graphics, where as it is used to obtain complex velocity potential in [2, 11]. Recently authors [7] applied it to obtain complex velocity potential of a flow of an ideal fluid in z-plane. In this paper we extend, the application of conformal mapping to construct a flow of an ideal fluid [1, 9] in different domain of z-plane.

2. PRELIMINARIES

Definition 2.1 (Conformal Mapping): *Let* $w = f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D. Then we say that $w = f(z)$ is **conformal** at z_0 if for every pair of smooth oriented curves C_1 and C_2 in D intersecting at z_0 the angle between C_1 and C_2 at z_0 is equal to the angle between the image curves C'_1 and C'_2 at $f(z_0)$ in both magnitude and sense. [11]

Corresponding Author: Mohammed Mukhtar Mohammed Zabih, Department of Mathematics, Maulana Azad College of Arts, Commerce and Science, Rouza Bagh, Aurangabad-431001.*

PRINCIPAL
Govt. College of Arts & Science
Aurangabad

Theorem 2.1: If f is an analytic function in a domain D containing z_0, and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0.[11]

Theorem 2.2: if $w = \Omega(z) = \phi(x,y) + i\,\psi(x,y)$ is a one-to-one conformal mapping of the domain D in the z-plane onto a domain D' in the w-plane such that the image of the boundary C of D is a horizontal line in the w-plane, then $f(z) = \overline{\Omega}'(z)$ is a complex representation of a flow of an ideal fluid in D.[11]

Theorem 2.3: The roots of polynomial p of the form $p(z) = z^n - c$, where c is a complex constant, are

$$z_{k+1} = |c|^{\frac{1}{n}}\left[\cos\left(\frac{\alpha + 2\pi k}{n}\right) + i\sin\left(\frac{\alpha + 2\pi k}{n}\right)\right]$$

where $k = 0, 1, \ldots, n-1$ and $\alpha = \arg c$; $0 \leqq \alpha < 2\pi$. [4]

Flow around a corner in wedge-shaped region in the first quadrant.

1. Construct a flow of an ideal fluid in the domain D between the lines $x = y$ & $y = 0$, in first quadrant.

Let the domain D be given by $\{(x,y)/x = y, y = 0 \text{ and } x > 0, y > 0\}$.

Consider the mapping $w = \Omega(z) = z^4$, defined on D.

If $z = x + iy$ and $w = u + iv$ then $u + iv = (x + iy)^4$, it implies that

$$u = (x^2 - y^2)^2 - 4x^2y^2;\ v = 4xy(x^2 - y^2) \quad (1)$$

The non-vanishing derivative of $w = \Omega(z)$, for $z \neq 0$ suggest that the mapping is conformal

We now show that the mapping is one-to-one by showing that $\Omega(z_1) = \Omega(z_2) \Rightarrow z_1 = z_2$

For this we use polar representation of $z \in D$. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, $0 < \theta_1 < \frac{\pi}{4}$ and $0 < \theta_2 < \frac{\pi}{4}$.

If $\Omega(z_1) = \Omega(z_2) \Rightarrow r_1^{\,4} e^{4i\theta_1} = r_2^{\,4} e^{4i\theta_2}$

Since, $0 < \theta_1 < \frac{\pi}{4}$ and $0 < \theta_2 < \frac{\pi}{4}$, it follows from above equation that $r_1 = r_2$ and $\theta_1 = \theta_2$. Therefore $z_1 = z_2$ and we conclude that $w = \Omega(z)$ is one-to-one on D.

If $w \in D'$, the image of domain D under $w = \Omega(z)$, then its pre-image $z = w^{\frac{1}{4}} \in D$. Note that among the four fourth roots of w only one corresponds to z (if $k = 0$ in theorem 2.3). Thus w is onto mapping.

Therefore $w = \Omega(z)$ is one-to-one conformal mapping from D onto D'.

If $x = y$ or $y = 0$, then in both cases, by second equation of (1) $v = 0$. Also $x > 0$, $y > 0$ and $x > y$ in D it implies that $v > 0$. Therefore the domain D and its boundary are mapped onto upper half plane, $Im\, w > 0$ and real axis, $v = 0$ (i.e horizontal line) of the w plane respectively.

Therefore, the stream lines of the flow are $v(x,y) = constant$ or $4xy(x^2 - y^2) = constant$, the rectangular hyperbolae having asymptotes $y = 0, x = y$.

Therefore, $f(z) = \overline{\Omega}'(z) = 4\,\bar{z}^3$ is a complex representation of a flow of an ideal fluid in D.

Flow around cylinder in a plane.

2. Construct a flow of an ideal fluid in the domain D, $|z + 1| > 1$ in the plane.

Consider the mapping $w = \Omega(z) = \frac{(1+i)z+2}{(1-i)z+2}$

$\Rightarrow u(x,y) = \frac{2x}{y^2+2(x+y+1)}$; $v(x,y) = \frac{x^2+y^2+2x}{y^2+2(x+y+1)}$

As the pole $-1-i$ of bilinear transformation $w = \Omega(z)$ lies on the boundary of D, therefore, it is analytic. Also $\Omega'(z) \neq 0$ implies that the mapping is conformal.
It can easily be shown that $w = \Omega(z)$ is one-to-one. Further, If $w \in D'$ then $z = \frac{2(w-1)}{(i-1)w+(1+i)} \in D$, hence onto mapping.

Therefore, the mapping $w = \Omega(z)$ is one-to-one, conformal mapping from D onto D'.

To find image of the domain D and its boundary, take $|z+1| = 1$

Therefore $|z+1| = \left|\frac{w(1+i)-(1-i)}{w(1-i)+(1+i)}\right| \Rightarrow 1 = \left|\frac{w(1+i)-(1-i)}{w(1-i)+(1+i)}\right| \Rightarrow 8v = 0 \Rightarrow v = 0$

If $|z+1| > 1 \Rightarrow \left|\frac{w(1+i)-(1-i)}{w(1-i)+(1+i)}\right| > 1 \Rightarrow 8v > 0 \Rightarrow v > 0$

Therefore, the domain D and its boundary are mapped onto upper half plane, $Im\, w > 0$ and real axis, $v = 0$ (i.e. horizontal line) of the w plane respectively

The stream lines of the flow are $v(x,y) = constant$ or $\frac{x^2+y^2+2x}{y^2+2(x+y+1)} = constant.$

Therefore $f(z) = \overline{\Omega}'(z) = \frac{-4i}{|(1+i)\bar{z}+2|^2}$ is a complex representation of fluid flow.

Flow about inclined plate

3. construct a flow of an ideal fluid in the domain above straight line, equally inclined to the co-ordinate axes in the z-plane.

Consider the linear map $w = \Omega(z) = z\, e^{\frac{-i\pi}{4}}$
$\Rightarrow u(x,y) = \frac{x+y}{\sqrt{2}}$; $v(x,y) = \frac{y-x}{\sqrt{2}}$
Clearly, $w = \Omega(z)$ is one-to-one, onto conformal mapping, as $z = w\, e^{\frac{i\pi}{4}} \in D$ for $\in D'$.

The boundary of D is the line $x = y$ in the z-plane which is mapped onto $v = 0$. In the domain D, $y > x \Rightarrow v > 0$

The stream lines of the flow are $v(x,y) = constant$ or $\frac{y-x}{\sqrt{2}} = constant$, which are straight lines incident to positive x-axis at angle $\frac{\pi}{4}$.
Therefore $\overline{\Omega}'(z) = e^{\frac{i\pi}{4}}$ is a complex representation of fluid flow. Clearly the flow is uniform.

CONCLUSION

Using conformal mapping, "complex velocity potential" of an ideal fluid can be directly determine.

REFERENCES

1. Bansal J.L. Viscous Fluid Dynamics. Oxford and India Book House Pvt Ltd. 2003
2. Chorlton F. Textbook of Fluid Dynamics. Cbs Publishers & Distributors, 2004
3. Churchill R. V. Brown J. and Verhey R. Complex variable and application. McGraw Hill Co. New York, (1974)
4. Coddington E.A. An Introduction to Ordinary Differential Equations. PHI, Private Limited New Delhi, 1999
5. Conway J.B. Functions of One Complex Variable I. Springer-Verlag New York, second edition.
6. Markushevich A.I. Complex Numbers and Conformal Mappings. Mir Publishers Moscow, 1982
7. Mohammed Mukhtar. Lahurikar R.M Applications of Conformal Mapping to Complex Velocity Potential of The Flow of An Ideal Fluid, International Journal of Mathematics Trends and Technology. Seventh Sense Research Group. Trichy. 52(3) (2017). P.P. 20-26
8. Donald T Piele.s Morris W. Firebaugh and Robert Manulik. Application to conformal mapping to potential theory through computer graphics. American Mathematical month. 1977. P.677-692.
9. Schlisetng H. Boundary layer theory. McGraw Hill Co. New York, (1968)
10. Ian N. Sneddon. Elements of Partial Differential Equations. McGraw- Hill Book Company. INC. Kosgakusha Company. LTD. Tokyo
11. Dennis G. Zill and Patrick D. Shanahan. A first Course in Complex Analysis with Applications. Second Edition. Jones and Bartlett Publishers.

Source of support: Nil, Conflict of interest: None Declared.